FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS
YEAR 12 HSC TASK 2 2015
ANSWERS COVER SHEET

Name: ____

QUESTION	MARK	H2	H3	H4	H5	H6	H7	H8	H9
Q1 – Q5	/5				✓				✓
Q6	/9				✓				✓
Q7	/13				✓	✓			✓
Q8	/11				✓				✓
Q9	/13				✓	✓			✓
Q10	/17				✓				
Q11	/9				✓			✓	
Q12	/13				✓				✓
TOTAL									
	/90				/90	/26		/12	/90

HSC Outcomes **Mathematics**

H2 constructs arguments to prove and justify results.

H3 manipulates algebraic expressions involving logarithmic and exponential functions.

H4 expresses practical problems in mathematical terms based on simple given models.

H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems.

H6 uses the derivative to determine the features of the graph of a function.

H7 uses the features of a graph to deduce information about the derivative.

H8 uses techniques of integration to calculate areas and volumes.

H9 communicates using mathematical language, notation, diagrams and graphs.

GIRRAWEEN HIGH SCHOOL
MATHEMATICS
Task 2

Year 12 Mathematics 2015

Time Allowed: 90 minutes

Instructions:
- There are 12 questions in this paper. All questions are compulsory.
- Start each question (6 – 12) on a new sheet of paper.
- Write on one side of the paper only.
- Show all necessary working.
- Board-approved calculators may be used.
- Marks may be deducted for careless or badly arranged work.

Questions 1 – 5 (5 marks)

Write the letter corresponding to the correct answer on your answer sheet.

1 What are the *x*-coordinates of the two turning points to the curve $f(x) = x^3 - 12x^2 + 36x + 10$?

(A) $x = -2,\ x = -6$

(B) $x = 2,\ x = 6$

(C) $x = 0,\ x = 3$

(D) $x = 3, x = 4$

2 The graph $y = f(x)$ passes through the point $(1, 4)$ and $f'(x) = 3x^2 - 2$.
Which of the following expressions is $f(x)$?

(A) $x^3 - 2x$

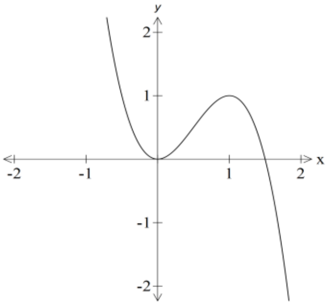
(B) $2x - 1$

(C) $x^3 - 2x + 3$

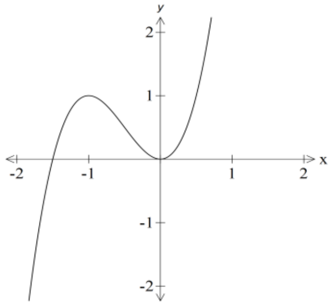
(D) $x^3 - 2x + 5$

3 Which of the following is the graph of $f(x) = 2x^3 - 3x^2$?

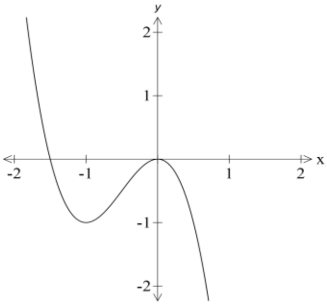
(A)



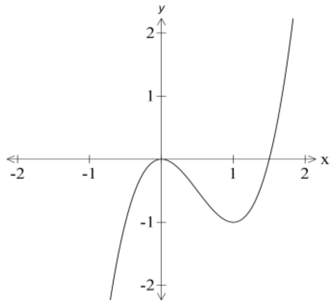
(B)



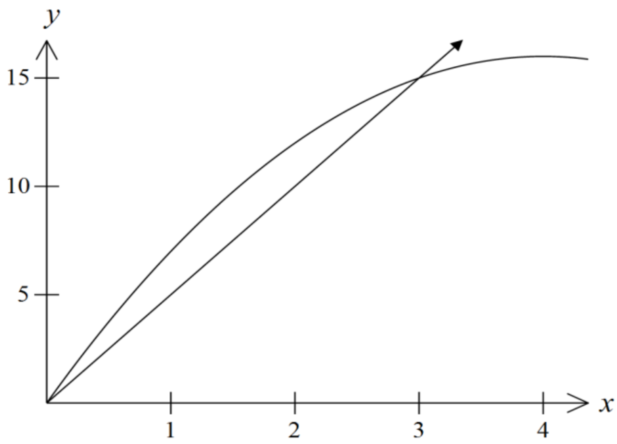
(C)



(D)



4 The diagram below shows the graph of $y = 5x$ and $y = 8x - x^2$.



What is the area between the curves $y = 5x$ and $y = 8x - x^2$?

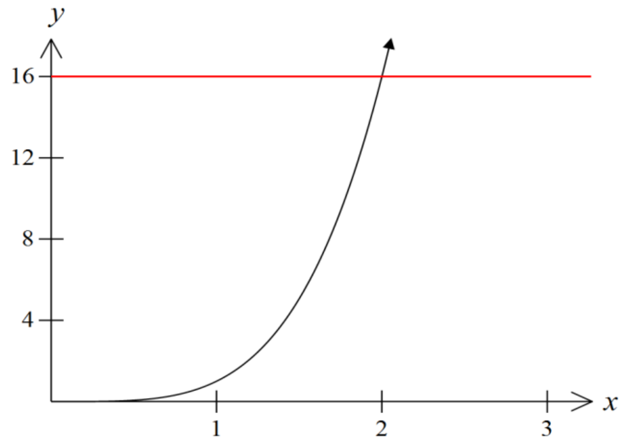
(A) 4.5 units^2

(B) 5.5 units^2

(C) 9.0 units^2

(D) 13.5 units^2

5 A region in the diagram is bounded by the curve $y = x^4$, the y-axis and the line $y = 16$.



Which of the following expressions is correct for the volume of the solid of revolution when this region is rotated about the y-axis?

(A) $V = \pi\int_0^2 x^8 dx$

(B) $V = \pi\int_0^{16} x^8 dx$

(C) $V = \pi\int_0^2 y^{\frac{1}{2}} dy$

(D) $V = \pi\int_0^{16} y^{\frac{1}{2}} dy$

For Questions 6 – 12, show all working. Start each question on a new sheet of paper.

Question 6 (9 marks)

a. The curve $y = 3x^2 + \dfrac{a}{x^2}$ has a turning point at $x = 3$. Find the value of a. **3**

b. If $y = 2x\sqrt{x}$, show that $\dfrac{y'}{y''} = 2x$. **3**

c. For what values of x is the curve $y = x^4 - 4x^3 - 18x^2$ concave up? **3**

Question 7 (13 marks)

a. Find the second derivative of $\dfrac{2x + 1}{2x - 1}$. **4**

b. For the function $f(x) = x^4 - 4x^3$,

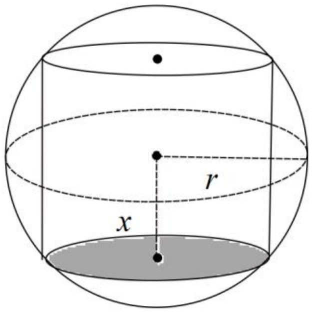
(i) Find the stationary points on the curve and determine their nature. **4**
(ii) Find any points of inflexion. **2**
(iii) Sketch the curve, showing all important features including the intercepts. **3**

Question 8 (11 marks)

a. The cost per hour of a bike ride is given by the formula

$C = x^2 - 15x + 70$ where x is the distance travelled in kilometres.
Find, using calculus, the distance that gives the minimum cost. **4**

b. A cylinder is to be made to fit inside a sphere of radius r cm as shown.



Let x be the distance of the base of the cylinder from the centre of the sphere.

(i) Find an expression for the radius of the base of the cylinder in terms of r and x. **1**

(ii) Show that the volume, V, of the cylinder is given by

$$V = 2\pi x(r^2 - x^2)$$ **2**

(iii) Find, in terms of r, the maximum volume of the cylinder. **5**

Question 9 (13 marks)

a. Find the primitive of:

(i) $\dfrac{x^4 - 3x + 4}{x^3}$ **3**

(ii) $\dfrac{1}{\sqrt{2x + 3}}$ **3**

b. The gradient function of a curve is $\dfrac{dy}{dx} = 3 - \dfrac{2}{x^2}$.
Find the equation of the curve if it passes through the point (1, -2). **3**

c. Find the equation of the curve $y = f(x)$, given that $\dfrac{d^2y}{dx^2} = 2x + 3$
and there is a minimum at (1, 3). **4**

Question 10 (17 marks)

a. Find:

(i) $\int (x^3 - 5x^2 + 7)dx$ **3**

(ii) $\int \left(\frac{1}{x^2} + \frac{2}{\sqrt{x}}\right) dx$ **3**

(iii) $\int x^2(x^2 + 3x - 4)dx$ **3**

b. Evaluate:

(i) $\int_1^9 \frac{t + 5}{\sqrt{t^3}} dt$ **4**

(ii) $\int_{-1}^0 (3x + 2)^3 dx$ **4**

Question 11 (9 marks)

a. Let $f(x) = \sqrt{25 - x^2}$.

(i) Copy and complete the table of values. **1**

x	0	1	2	3	4	5
$f(x)$						

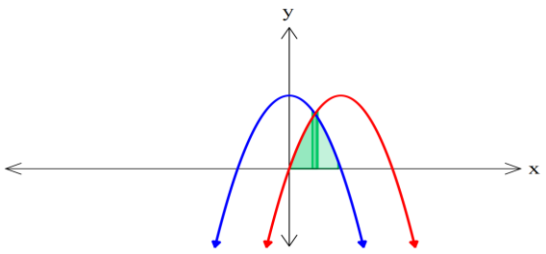
(ii) Use the Trapezoidal Rule with these function values to find an approximation for $\int_0^5 \sqrt{25 - x^2} dx$ correct to 3 decimal places. **3**

b. Use 2 applications of Simpson's Rule to find an approximate value for the area bounded by the curve $y = \frac{4}{x}$, the x - axis and the lines $x = 1$ and $x = 2$. **5**

Question 12 (13 marks)

a. Find the area between the curve $y = x^2 - 5x + 4$ and the x - axis between $x = 2$ and $x = 5$. **4**

b. The shaded area is enclosed between the parabolas $f(x) = 4 - x^2, g(x) = -x^2 + 4x$ and the x - axis. Find the shaded area. **4**



c. The area enclosed between the parabola $f(x) = x^2 + 2$ and the line $g(x) = 4x - 1$ is rotated about the y – axis. Find the volume of the solid generated. **5**

End of Examination

YEAR 12 MATHEMATICS TASK 2 2015
SOLUTIONS

MC (5 marks)

1. $f(x) = x^3 - 12x^2 + 36x + 10$

$f'(x) = 3x^2 - 24x + 36$

TP $\Rightarrow f'(x) = 0$

$3(x^2 - 8x + 12) = 0$

$(x-2)(x-6) = 0$

$x = 2, 6$ **B**

2. $f'(x) = 3x^2 - 2$

$f(x) = x^3 - 2x + c$

passes through $(1, 4)$

ie. $1^3 - 2(1) + c = 4$

$c = 5$

$\therefore f(x) = x^3 - 2x + c$ **D**

3. $f(x) = 2x^3 - 3x^2$

x-int $\Rightarrow x^2(2x-3) = 0$

$x = 0, \frac{3}{2}$

↑ double roots

SP $\Rightarrow f'(x) = 6x^2 - 6x = 0$

$6x(x-1) = 0$

$x = 0, 1$

$f''(x) = 12x - 6$

At $(0,0)$, $f''(x) = -6$ $\therefore$ max

At $(1, -1)$, $f''(x) = 6$ $\therefore$ min **D**

4. $A = \int_0^3 (8x - x^2 - 5x)\,dx$

$= \int_0^3 (3x - x^2)\,dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$ **A**

$= 4\frac{1}{2}$

5. $y = x^4 \Rightarrow x^2 = y^{1/2}$

$V = \pi \int_0^{11} y^{1/2}\,dy$ **D**

Question 6 (9 marks)

a) $y = 3x^2 + \frac{a}{x^2}$

$= 3x^2 + ax^{-2}$

$\frac{dy}{dx} = 6x - \frac{2a}{x^3}$

TP $\Rightarrow \frac{dy}{dx} = 0$

$6(3) - \frac{2a}{3^3} = 0$

$\frac{2a}{27} = 18$

$a = 243$ (3)

b) $y = 2x\sqrt{x}$

$= 2x^{3/2}$

$\frac{dy}{dx} = 3x^{1/2}$; $\frac{d^2y}{dx^2} = \frac{3}{2}x^{-1/2}$

$= 3\sqrt{x}$ $\qquad = \frac{3}{2\sqrt{x}}$

$\frac{y'}{y''} = 3\sqrt{x} \times \frac{2\sqrt{x}}{3}$

$= 2x$ (3)

c) $y = x^4 - 4x^3 - 18x^2$

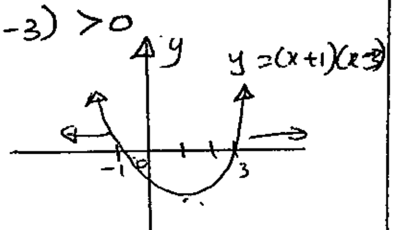
$\frac{dy}{dx} = 4x^3 - 12x^2 - 36x$

$\frac{d^2y}{dx^2} = 12x^2 - 24x - 36$

Concave up $\Rightarrow \frac{d^2y}{dx^2} > 0$

$12(x^2 - 2x - 3) > 0$

$(x+1)(x-3) > 0$



Concave up when $x < -1$, $x > 3$

(3)

Question 7 (13 marks)

a) $y = \frac{2x+1}{2x-1}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$

$= \frac{-4}{(2x-1)^2} = -4(2x-1)^{-2}$

$\frac{d^2y}{dx^2} = 8(2x-1)^{-3}.2$

$= \frac{16}{(2x-1)^3}$ (4)

b) $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$; $f''(x) = 12x^2 - 24x$

i) SP $\Rightarrow f'(x) = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0, 3$

when $x = 3$, $f''(3) = 12(3)^2 - 24(3)$

$= 36 > 0$

$\therefore$ minimum at $(3, -27)$

when $x = 0$, $f''(0) = 0$

$\therefore$ possible point of inflexion

Test for Point of inflexion

x	-1	0	1
$f''(x)$	36	0	-12

there is a change in concavity

$\therefore$ Horizontal point of inflexion at $(0,0)$ (4)

ii) Points of inflexion $\Rightarrow f''(x) = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$, $x = 2$

Horizontal point of inflexion at $(0,0)$ — shown in (i)

Test $x = 2$

x	1·5	2	2·5
$f''(x)$	-9	0	15

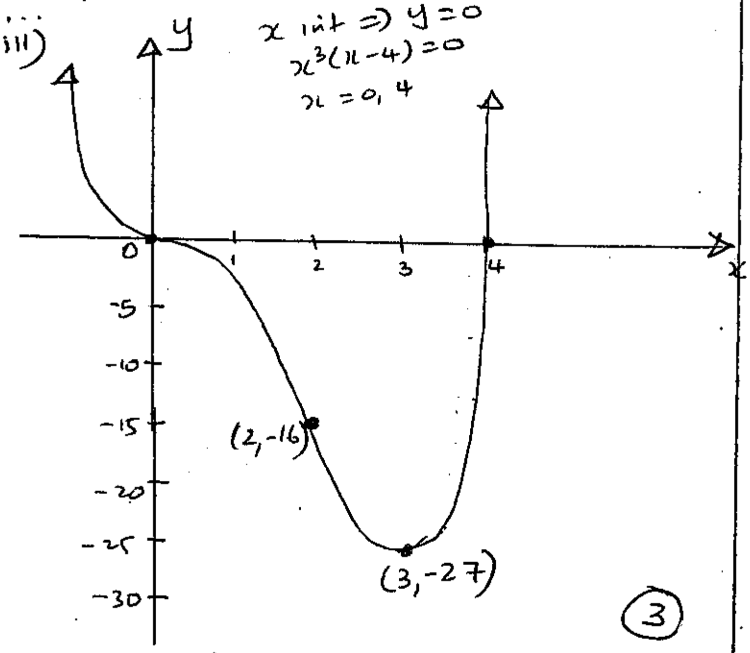
there is a change in concavity

$\therefore$ point of inflexion at $(2, -16)$ (2)

iii) x int $\Rightarrow y = 0$

$x^3(x-4) = 0$

$x = 0, 4$



(3)

Question 8 (11 marks)

a) $C = x^2 - 15x + 70$

$\frac{dC}{dx} = 2x - 15$; $\frac{d^2C}{dx^2} = 2$

For minimum cost, $\frac{dC}{dx} = 0$

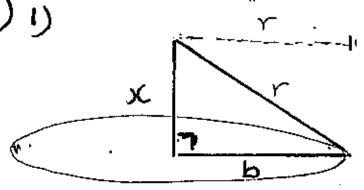
ie. $2x - 15 = 0$

$x = 7\frac{1}{2}$

When $x = 7\frac{1}{2}$, $\frac{d^2C}{dx^2} > 0$

$\therefore$ Minimum when $x = 7\frac{1}{2}$ km (4)

8. b) i) Let radius of base of cylinder = b



$$b^2 = r^2 - x^2$$

$$b = \sqrt{r^2 - x^2}$$ ①

ii) Height of cylinder = $2x$

$$V = \pi r^2 h$$

$$= \pi \left(\sqrt{r^2 - x^2}\right)^2 (2x)$$

$$= 2\pi x (r^2 - x^2)$$ ②

iii) $V = 2\pi r^2 x - 2\pi x^3$

$$\frac{dV}{dx} = 2\pi r^2 - 6\pi x^2$$

maximum V when $\frac{dV}{dx} = 0$

$$2\pi r^2 - 6\pi x^2 = 0$$

$$2\pi(r^2 - 3x^2) = 0$$

$$r^2 = 3x^2$$

$$x = \frac{r}{\sqrt{3}} \quad (\text{since } x > 0)$$

$$\frac{d^2V}{dx^2} = -12\pi x$$

when $x = \frac{r}{\sqrt{3}}$, $\frac{d^2V}{dx^2} = \frac{-12\pi r}{\sqrt{3}} < 0$

$\therefore$ maximum when $x = \frac{r}{\sqrt{3}}$

Maximum Volume

$$= 2\pi \frac{r}{\sqrt{3}} \left(r^2 - \frac{r^2}{3}\right)$$

$$= \frac{4\pi r^3}{3\sqrt{3}} \text{ cm}^3$$ ④

Question 9 (13 marks)

a) i) $y' = \frac{x^4 - 3x + 4}{x^3}$

$$= x - \frac{3}{x^2} + \frac{4}{x^3}$$

$$= x - 3x^{-2} + 4x^{-3}$$

$$y = \frac{x^2}{2} + 3x^{-1} - \frac{4x^{-2}}{2} + C$$

$$= \frac{x^2}{2} + \frac{3}{x} - \frac{2}{x^2} + C$$ ③

ii) $y' = \frac{1}{\sqrt{2x+3}}$

$$= (2x+3)^{-1/2}$$

$$y = \frac{(2x+3)^{1/2}}{\frac{1}{2} \times 2} + C$$

$$y = \sqrt{2x+3} + C$$ ③

b) $\frac{dy}{dx} = 3 - \frac{2}{x^2}$

$$= 3 - 2x^{-2}$$

$$y = 3x + 2x^{-1} + C$$

$$y = 3x + \frac{2}{x} + C$$

when $x = 1$, $y = -2$

$$-2 = 3(1) + \frac{2}{1} + C$$

$$C = -7$$

$\therefore y = 3x + \frac{2}{x} - 7$ ③

c) $\frac{d^2y}{dx^2} = 2x + 3$

$$\frac{dy}{dx} = x^2 + 3x + C$$

Minimum at $(1, 3)$

i.e. when $x = 1$, $\frac{dy}{dx} = 0$

$$(1)^2 + 3(1) + C = 0$$

$$C = -4$$

$$\therefore \frac{dy}{dx} = x^2 + 3x - 4$$

9) con't

c) $y = \frac{x^3}{3} + \frac{3x^2}{2} - 4x + c$

passes through $(1,3)$

$\therefore \frac{1}{3} + \frac{3}{2} - 4 + c = 3$

$c = 5\frac{1}{6}$

$\therefore y = \frac{x^3}{3} + \frac{3x^2}{2} - 4x + 5\frac{1}{6}$ (4)

Question 10 (17 marks)

a) i) $\int (x^3 - 5x^2 + 7)\,dx$

$= \frac{x^4}{4} - \frac{5x^3}{3} + 7x + C$ (3)

ii) $\int \left(\frac{1}{x^2} + \frac{2}{\sqrt{x}}\right) dx$

$= \int (x^{-2} + 2x^{-1/2})\,dx$

$= -x^{-1} + 4x^{1/2} + c$

$= -\frac{1}{x} + 4\sqrt{x} + C$ (3)

iii) $\int x^2(x^2 + 3x - 4)\,dx$

$= \int (x^4 + 3x^3 - 4x^2)\,dx$

$= \frac{x^5}{5} + \frac{3x^4}{4} - \frac{4x^3}{3} + C$ (3)

b) i) $\int_1^9 \frac{t+5}{\sqrt{t^3}}\,dt$

$= \int_1^9 \frac{t+5}{t^{3/2}}\,dt$

$= \int_1^9 \left(t^{-1/2} + 5t^{-3/2}\right) dt$

$= \left[2\sqrt{t} - \frac{10}{\sqrt{t}}\right]_1^9$

$= 6 - \frac{10}{3} - (-8)$

$= 10\frac{2}{3}$ (4)

ii) $\int_{-1}^{0} (3x+2)^3\,dx$

$= \left[\frac{(3x+2)^4}{12}\right]_{-1}^{0}$

$= \frac{1}{12}(2^4 - 1^4)$

$= \frac{15}{12}$

$= \frac{5}{4}$ (4)

Question 11 (9 marks)

a) i) $f(x) = \sqrt{25 - x^2}$

	1	2	2	2	2	1
x	0	1	2	3	4	5
$f(x)$	5	4.899	4.583	4	3	0

$h = 1$ (1)

ii) $\int_0^5 \sqrt{25 - x^2}\, dx \doteq \frac{1}{2}\left\{5 + 2(4.899 + 4.583 + 4 + 3) + 0\right\}$

$\doteq 18.982$ (3)

b)

	1	4	2	4	1
x	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
$f(x)$	4	$\frac{16}{5}$	$\frac{8}{3}$	$\frac{16}{7}$	2

$h = 0.25$

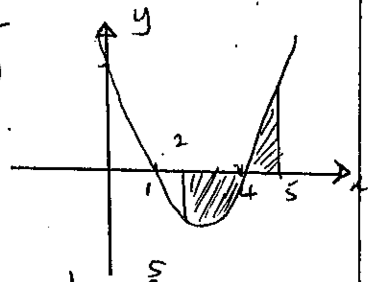
$A \doteq \frac{0.25}{3}\left\{4 + 4\left(\frac{16}{5} + \frac{16}{7}\right) + 2\left(\frac{8}{3}\right) + 2\right\}$

$\doteq 2\frac{487}{630}$ (5)

Question 12 (13 marks)

a) $y = x^2 - 5x + 4$

$= (x-1)(x-4)$



$A = \left| \int_2^4 (x^2 - 5x + 4)\,dx \right| + \int_4^5 (x^2 - 5x + 4)\,dx$

$= \left| \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_2^4 \right| + \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x \right]_4^5$

$= \left| \left(\frac{4^3}{3} - \frac{5(4)^2}{2} + 4(4) \right) - \left(\frac{2^3}{3} - \frac{5(2)^2}{2} + 4(2) \right) \right|$

$+ \left(\left[\frac{5^3}{3} - \frac{5(5)^2}{2} + 4(5) \right] - \left[\frac{4^3}{3} - \frac{5(4)^2}{2} + 4(4) \right] \right)$

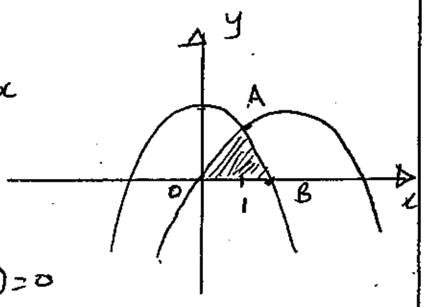
$= 3\frac{1}{3} + 1\frac{5}{6}$

$= 5\frac{1}{6}$ square units (4)

b)

A: $4 - x^2 = -x^2 + 4x$

$x = 1$



B: $f(x) = 4 - x^2$

x int $\Rightarrow (2-x)(2+x) = 0$

$x = \pm 2$

$\therefore B(2, 0)$

$A = \int_0^1 (4x - x^2)\,dx + \int_1^2 (4 - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3} \right]_0^1 + \left[4x - \frac{x^3}{3} \right]_1^2$

$= \left(2 - \frac{1}{3} - 0 \right) + \left(\left(8 - \frac{8}{3} \right) - \left(4 - \frac{1}{3} \right) \right)$

$= \frac{5}{3} + \frac{5}{3}$

$= 3\frac{1}{3}$ square units

c) $f(x) = x^2 + 2$; $g(x) = 4x - 1$

Points of Intersection

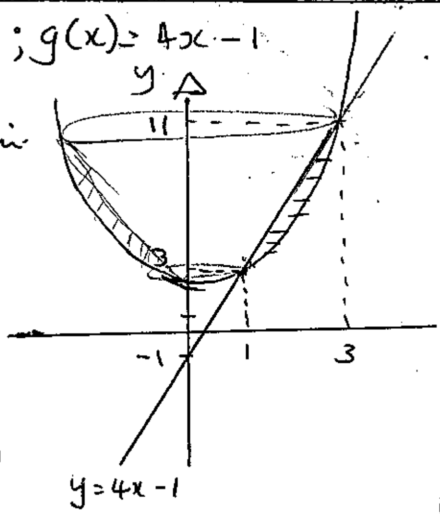
$x^2 + 2 = 4x - 1$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, 3$

pts $(1, 3), (3, 11)$



$y = x^2 + 2$

$x^2 = y - 2$

$y = 4x - 1$

$x = \frac{y+1}{4}$

$x^2 = \frac{(y+1)^2}{16}$

$V = \pi \int_3^{11} (y - 2)\,dy - \pi \int_3^{11} \frac{(y+1)^2}{16}\,dy$

$= \pi \int_3^{11} \left(y - 2 - \frac{(y+1)^2}{16} \right) dy$

$= \pi \left[\frac{y^2}{2} - 2y - \frac{(y+1)^3}{48} \right]_3^{11}$

$= \pi \left[\left(\frac{11^2}{2} - 2(11) - \frac{12^3}{48} \right) - \left(\frac{3^2}{2} - 2(3) - \frac{4^3}{48} \right) \right]$

$= \frac{16\pi}{3}$ cubic units.

(5)